Section 6.4: Factoring and Solving Polynomial Functions

Factoring Review: Factor each of the polynomial expressions.

1. \(x^2 - 21x + 20\)
   \((x-20)(x-1)\)

2. \(9m^2 - 36n^4\)
   \((3m-4n^2)(3m+4n^2)\)

3. \(4x^2 - 36x + 81\)
   \((2x-9)^2\)

4. \(2x^2 + 11x - 21\)
   \(a \cdot c = -42\)
   \(b = -42\)
   \(14, -3\)

5. \(4x^2 - 24x\)
   \(4x(x-6)\)

6. \(6x^2 + x - 15\)
   \(a \cdot c = -90\)
   \(b = -90\)
   \(\pm 1\)
   \(10, -9\)
   \(2x(3x+5) - 3(3x+5)\)
   \((3x+5)(2x-3)\)

In #7-9, solve the equation for \(x\).

7. \(x^2 = -12x - 36\)
   \(x^2 + 12x + 36 = 0\)
   \((x+6)^2 = 0\)
   \(x = -6\)

8. \(6x^2 + 13x = 5\)
   \(6x^2 + 15x - 2x - 5 = 0\)
   \(3x(2x+5) - 1(2x+5) = 0\)
   \((2x+5)(3x-1) = 0\)
   \(x = \frac{5}{3}, \frac{1}{3}\)

9. \(4x^2 - 100 = 0\)
   \(4(x^2 - 25) = 0\)
   \(4(x-5)(x+5) = 0\)
   \(x = \pm 5\)
Methods of Factoring/Factoring Patterns that should be familiar:

- Difference of Squares
- Perfect Square Trinomials
- General Trinomials (using short-cut method/British Method)

There will be new factoring patterns and processes now that the powers of $x$ are increasing. (We are working with polynomials now, not just quadratics.)

**Two Special Factoring Patterns**

(1) Sum of Two Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

(2) Difference of Two Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Examples) Factor each of the following expression.

1. $x^3 + 8 = (x)^3 + (2)^3 = (x + 2)(x^2 - 2x + 4)$

2. $64 + 27x^3 = (8)^3 + (3x)^3 = (8 + 3x)(64 - 24x + 9x^2)$

3. $8x^3 - 1 = (2x)^3 - (1)^3 = (2x - 1)(4x^2 + 2x + 1)$

4. $343a^3 + 27b^3 = (7a)^3 + (3b)^3 = (7a + 3b)(49a^2 - 21ab + 9b^2)$

5. $16u^5 - 250u^2 = 2u^2(8u^3 - 125) = 2u^2((2u)^3 - (5)^3) = 2u^2(2u - 5)(4u^2 + 10u + 25)$

6. $216x^3y^3 - 1 = (6xy)^3 - (1)^3 = (6xy - 1)(36x^2y^2 + 6xy + 1)$
**When factoring using a sum/difference of two cubes, the trinomial in the factoring pattern is always unfactorable.**

**Common Cubes** (to look out for)

\[
\begin{align*}
1^3 &= 1 \\
2^3 &= 8 \\
3^3 &= 27 \\
4^3 &= 64 \\
5^3 &= 125 \\
6^3 &= 216 \\
7^3 &= 343 \\
8^3 &= 512 \\
9^3 &= 729 \\
10^3 &= 1000
\end{align*}
\]

**Four-Term Polynomials:** Factor by Grouping.

- Expression should be in standard form before factoring
- Group pairs of terms and take out the GCF of each group
- Always check at the end to make sure the expression is completely factored

Examples) Factor each of the following.

1. \(x^3 - 2x^2 - 9x - 18\)
   \[
   = (x^2 - 9)(x + 2)
   = (x - 3)(x + 3)(x + 2)
   \]
2. \(x^3 + 3x^2 + 10x + 30\)
   \[
   = (x^2 + 10)(x + 3)
   = (x + 5)(x + 3)(x + 2)
   \]
3. \(3x^3 - 6x^2 + x - 2\)
   \[
   = (3x^2 - 1)(x - 2)
   \]
4. \(x^2y^2 - 3x^2 - 4y^2 + 12\)
   \[
   = (y^2 - 3)(x^2 - 4)
   = (y - 3)(y + 3)(x - 2)(x + 2)
   \]
**Quadratic Form:** $au^2 + bu + c$, where $u$ is an expression of $x$

**Examples:***

1. $x^4 - 5x^2 + 6$
   \[ (x^2)^2 - 5(x^2) + 6 \]
   \[ (x^2 - 3)(x^2 - 2) \]

2. $4x^4 - 5x^2 - 9$
   \[ 4(x^2)^2 - 5(x^2) - 9 \]
   \[ 4u^2 - 5u - 9 \]
   \[ 4u^2 + 4u - 9u - 9 \]
   \[ u^2 - 9 \]
   \[ (u - 3)(u + 3) \]
   \[ (x^2 - 3)(x^2 + 3) \]

3. $x^6 + 10x^3 + 16$
   \[ (x^3)^2 + 10(x^3) + 16 \]
   \[ (x^3 + 8)(x^3 + 2) \]
   \[ (x + 2)(x^2 - 2x + 4)(x^2 + 2) \]

**Practice:** Factor and/or solve each expression or equation below. (solve for imaginary solutions as well, if applicable!)

1. $x^6 + 3x^3 + 2$
   \[ (x^3 + 2)(x^3 + 1) \]
   \[ (x + 2)(x + 1)(x^2 - x + 1) \]

2. $27x^3 + 216$
   \[ 27(x^3 + 8) \]
   \[ 27(x + 3)(x^2 - 2x + 4) \]

3. $x^3 + 3x^2 - 4x - 12 = 0$
   \[ x^2(x + 3) - 4(x + 3) = 0 \]
   \[ (x + 3)(x^2 - 4) = 0 \]
   \[ (x + 3)(x - 2)(x + 2) = 0 \]
   \[ x = -3, 2, -2 \]

4. $125x^3 = 8$
   \[ 125x^3 - 8 = 0 \]
   \[ (5x - 2)(25x^2 + 10x + 4) = 0 \]
   \[ 5x - 2 = 0 \]
   \[ x = \frac{2}{5} \]
   \[ 7\text{ quadratic formula!} \]
   \[ x = \frac{2 \pm \sqrt{4 - 4 \cdot 2 \cdot 5}}{2 \cdot 2} \]
   \[ x = \frac{2 \pm \sqrt{-100}}{4} \]
   \[ x = \frac{2 \pm 10i}{4} \]
   \[ x = \frac{1}{2} \pm \frac{5}{2}i \]
   \[ x = 0.5 \pm 1.25i \]
Solving Polynomial Equations by Factoring:

The Zero Product Property can be extended to solve equations with polynomials of higher degrees.

1. Factor the expression completely.
2. Set each factor equal to zero and solve. There may be imaginary solutions.

Examples) Solve each equation.

1. \(2x^5 + 24x = 14x^3\)
   \[\begin{align*}
   &\quad \quad ax^5 - 14x^3 + 24x = 0 \\
   &ax(x^4 - 7x^2 + 12) = 0 \\
   &ax(x^2 - 3)(x^2 - 4) = 0 \\
   &ax(x^2 - 3)(x - 2)(x + 2) = 0 \\
   &x = 0 \quad x = \pm \sqrt{3} \quad x = \pm 2
   \end{align*}\]

2. \(2x^5 - 18x = 0\)
   \[\begin{align*}
   &\quad \quad ax(x^4 - 9) = 0 \\
   &ax(x^2 - 3)(x^2 + 3) = 0 \\
   &ax = 0 \quad x^2 - 3 = 0 \quad x^2 + 3 = 0 \\
   &x = 0 \quad x = \pm \sqrt{3} \quad x = \pm \sqrt{3}i
   \end{align*}\]

3. \(3x^7 = 81x^4\)
   \[\begin{align*}
   &\quad \quad 3x^7 - 81x^4 = 0 \\
   &3x^4(x^3 - 27) = 0 \\
   &3x^4(x - 3)(x^2 + 3x + 9) = 0 \\
   &x = 0 \quad x = 3 \quad \text{quad. formula} \quad x = \frac{-3 \pm \sqrt{9 - 4(1)(9)}}{2} \\
   &x = -3 \pm \sqrt{9 - 36} \\
   &x = -3 \pm 3i \sqrt{3}
   \end{align*}\]

4. \(x^3 - 4x^2 + 4x - 16 = 0\)
   \[\begin{align*}
   &\quad \quad x^2(x - 4) + 4(x - 4) = 0 \\
   &x(x - 4)(x^2 + 4) = 0 \\
   &x = 4 \quad x^2 + 4 = 0 \\
   &x = 4 \quad x = \pm 2i
   \end{align*}\]

5. \(125x^3 = 216\)
   \[\begin{align*}
   &\quad \quad 125x^3 - 216 = 0 \\
   &5x^3 - 6 = 0 \\
   &x = \frac{6}{5} \quad \text{quad. formula} \quad x = -\frac{30 \pm \sqrt{900 - 4(25)(36)}}{2(25)} \\
   &x = -\frac{30 \pm \sqrt{900 - 3600}}{50} \quad x = -\frac{30 \pm 90 - 360}{50} \quad x = -\frac{30 \pm 30i \sqrt{3}}{50} \quad x = -\frac{3 \pm 3i \sqrt{3}}{2}
   \end{align*}\]

6. \(1000x^3 + 27 = 0\)
   \[\begin{align*}
   &\quad \quad 1000x^3 = -27 \\
   &x = -\frac{3}{10} \quad \text{quad. formula} \quad x = -\frac{3 \pm \sqrt{9 - 4(1000)(27)}}{2} \quad x = -\frac{3 \pm \sqrt{-27900}}{2}
   \end{align*}\]
7. \[x^3 + 8x^2 = -16x\]
\[x^3 + 8x^2 + 16x = 0\]
\[x(x^2 + 8x + 16) = 0\]
\[x(x + 4)^2 = 0\]
\[x = 0, -4\]